

# THE BASS AND TOPOLOGICAL STABLE RANKS FOR ALGEBRAS OF ALMOST PERIODIC FUNCTIONS ON THE REAL LINE

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ABSTRACT. Let  $\Lambda$  be a sub-semigroup of the reals. We show that the Bass and topological stable ranks of the algebras  $AP_\Lambda = \{f \in AP : \sigma(f) \subseteq \Lambda\}$  of almost periodic functions on the real line and with Bohr spectrum in  $\Lambda$  are infinite whenever the algebraic dimension of the  $\mathbb{Q}$ -vector space generated by  $\Lambda$  is infinite. This extends Suárez's result for  $AP_{\mathbb{R}} = AP$ . Also considered are general subalgebras of  $AP$ .

## INTRODUCTION

Let  $C_b(\mathbb{R}, \mathbb{C})$  denote the set of bounded, continuous functions on  $\mathbb{R}$  with values in  $\mathbb{C}$  and let  $AP$  be the uniform closure in  $C_b(\mathbb{R}, \mathbb{C})$  of the set of all functions of the form

$$Q(t) := \sum_{j=1}^N a_j e^{i\lambda_j t},$$

where  $a_j \in \mathbb{C}$ ,  $\lambda_j \in \mathbb{R}$  and  $N \in \mathbb{N}^*$ .  $AP$  is the set of *almost periodic functions*. We call  $Q$  a *generalized trigonometric polynomial*. Under the usual pointwise algebraic operations,  $AP$  is a point separating function algebra on  $\mathbb{R}$  with the property that  $f \in AP$  implies that  $\bar{f} \in AP$ . Harald Bohr developed the basic theory for this space in a series of papers [2, 3]. We also refer to the nice books by Corduneanu [8] and Besicovich [1] for an introduction into this important class of functions. Modern treatments and applications to operator theory can be found for example in [5].

In our paper we are interested in a specific algebraic property and its topological counterpart for subalgebras of  $AP$ : namely the Bass and topological stable ranks (see below for the definition). In [16] Daniel Suárez showed that these ranks are infinite for  $AP$ . This result reflected again the close connection of  $AP$  to the infinite polydisk algebra  $A(\mathbb{D}^\infty)$ , a connection unveiled by Bohr in his fundamental papers. Let us mention that the stable ranks of  $A(\mathbb{D}^\infty)$  were known to be infinite (see for instance [12]). Suárez's approach did not allow to calculate the Bass stable rank of the analytic trace of  $AP$ , namely the algebra

$$AP^+ = \{f \in AP : \widehat{f}(\lambda) = 0 \text{ for all } \lambda \in ]-\infty, 0[ \}$$

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of those almost periodic functions on  $\mathbb{R}$ <sup>1</sup> that admit a bounded holomorphic extension to the upper half-plane. Indeed, he deduced the fact that  $\text{bsr AP} = \infty$  from his Theorem that the topological stable rank of AP is infinite and Rieffel's result that for commutative  $C^*$ -algebras the Bass stable rank coincides with the topological one (see [15]). Since in  $\text{AP}^+$  the only functions  $f$  satisfying  $f \in \text{AP}^+$  and  $\bar{f} \in \text{AP}^+$  are the constants, Suárez's method cannot be used and adapted to handle the algebra  $\text{AP}^+$ .

Using a different and actually more elementary method than Suárez's, we extend his result to the following natural subalgebras of AP: for a sub-semigroup  $\Lambda$  of  $(\mathbb{R}, +)$  let

$$\text{AP}_\Lambda = \{f \in \text{AP} : \sigma(f) \subseteq \Lambda\},$$

where  $\sigma(f)$  denotes the Bohr spectrum of  $f \in \text{AP}$  (see below). Our main result tells us that the Bass and topological stable ranks of  $\text{AP}_\Lambda$  are infinite as well, whenever the dimension of the vector space  $[\Lambda]$  generated by  $\Lambda$  over  $\mathbb{Q}$  is infinite. Moreover, if this latter condition is not satisfied, then these stable ranks can be finite. We also consider more general subalgebras of AP.

Finally, we would like to mention that a first attempt to calculate the stable ranks for  $\text{AP}_\Lambda$  and, in particular, the one for  $\text{AP}^+$ , was given by Mikkola and Sasane in [11].

## 1. THE TOOLS FOR OUR PROOF

For the reader's convenience, we state here several results, surely known to people working with AP functions, that are necessary to understand our proofs of the new results. The most basic tool will be Kronecker's approximation theorem. A very elegant proof can be found in [10].

**Theorem 1.1** (Kronecker). *The following statements are true:*<sup>2</sup>

(C<sub>N</sub>) *For  $j = 1, \dots, N$ , let  $\lambda_j \in \mathbb{R}$ . Suppose that  $\{\lambda_1, \dots, \lambda_N\}$  is linearly independent over  $\mathbb{Q}$ . Then*

$$C := \{(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t}) : t \in \mathbb{R}\}$$

*is dense in  $\mathbb{T}^N$ .*

(D<sub>N</sub>) *For  $j = 1, \dots, N$ , let  $\lambda_j \in \mathbb{R}$ . Suppose that  $\{\lambda_1, \dots, \lambda_N, 2\pi\}$  is linearly independent over  $\mathbb{Q}$ . Then*

$$D := \{(e^{i\lambda_1 n}, \dots, e^{i\lambda_N n}) : n \in \mathbb{N}\}$$

*is dense in  $\mathbb{T}^N$ .*

Let us recall the definitions of the fundamental notions in connection with almost periodic functions (see for instance [8]).

**Definition 1.2.** Let  $f \in \text{AP}$ . If  $\lambda \in \mathbb{R}$ , the associated *Fourier-Bohr coefficient*  $\widehat{f}(\lambda)$  is defined as

$$\widehat{f}(\lambda) = \lim_{|I| \rightarrow \infty} \frac{1}{|I|} \int_I f(t) e^{-i\lambda t} dt,$$

where  $I$  runs through the set of all compact intervals in  $\mathbb{R}$ .

<sup>1</sup>Here  $\widehat{f}(\lambda)$  is the Fourier-Bohr coefficient associated with  $\lambda$ ; see below for the Definition.

<sup>2</sup> Here [C] stands for continuous, and [D] for discrete.

**Proposition 1.3.** *If  $f \in \text{AP}$ , then  $\widehat{f}(\lambda)$  exists for every  $\lambda \in \mathbb{R}$  and  $\widehat{f}(\lambda) \neq 0$  for at most a countable number of  $\lambda$ .*

**Definition 1.4.** If  $f \in \text{AP}$ , then the *Bohr-spectrum*,  $\sigma(f)$ , of  $f$  is the set of all  $\lambda \in \mathbb{R}$  for which the associated Fourier-Bohr coefficient  $\widehat{f}(\lambda)$  is not zero. If  $\sigma(f) = \{\lambda_n : n \in I\}$ ,  $I \subseteq \mathbb{N}$ , then the *Fourier-Bohr series* associated with  $f$  is the formal series

$$f \sim \sum_{n \in I} \widehat{f}(\lambda_n) e^{i\lambda_n t}.$$

The main tool will be the following approximation theorem (see [8] for a classroom proof and [5] for a shorter, but more advanced proof).

**Theorem 1.5.**

- (1) *The Fourier-Bohr series uniquely determines  $f$  whenever  $f \in \text{AP}$ .*
- (2) *Let  $f$  be an almost periodic function on  $\mathbb{R}$  with Bohr spectrum  $\sigma(f)$ . Then there exists a sequence  $(q_n)$  of generalized trigonometric polynomials with  $\sigma(q_n) \subseteq \sigma(f)$  converging uniformly to  $f$ .*

## 2. SOME ALGEBRAIC PROPERTIES OF $\text{AP}$ AND $\text{AP}_\Lambda$

We first present a simple proof of the well-known “corona theorem” for the algebra of almost periodic functions.

**Definition 2.1.** Let  $A$  be a commutative unital algebra (real or complex) (or just a commutative unital ring) with identity element denoted by 1. An  $n$ -tuple  $(f_1, \dots, f_n) \in A^n$  is said to be *invertible* (or *unimodular*), if there exists  $(x_1, \dots, x_n) \in A^n$  such that the Bézout equation  $\sum_{j=1}^n x_j f_j = 1$  is satisfied. The set of all invertible  $n$ -tuples is denoted by  $U_n(A)$ . Note that  $U_1(A) = A^{-1}$ , the latter being the set of invertible elements in  $A$ .

**Theorem 2.2.** *The following assertions hold:*

- i) *An element  $F \in \text{AP}$  is invertible if and only if  $\delta := \inf_{\mathbb{R}} |F| > 0$ .*
- ii)  *$U_n(\text{AP}) = \{(F_1, \dots, F_n) \in \text{AP}^n : \inf_{\mathbb{R}} \sum_{j=1}^n |F_j| > 0\}$ .*

*Proof.* i) Let  $F \in U_1(\text{AP})$ . Then  $F$  is invertible in  $C_b(\mathbb{R}, \mathbb{C})$  and so  $F$  is bounded away from zero. Conversely, let  $F \in \text{AP}$  satisfy  $|F| \geq \delta > 0$  on  $\mathbb{R}$ . Since  $F$  is bounded, there exists  $M > 0$  such that  $\delta \leq |F| \leq M$  on  $\mathbb{R}$ . By Weierstrass’ approximation theorem, let  $(p_n)$  be a sequence of polynomials in two variables such that

$$\sup_{\delta \leq |z| \leq M} \left| p_n(z, \bar{z}) - \frac{1}{z} \right| \rightarrow 0.$$

Then

$$\sup_{x \in \mathbb{R}} \left| p_n\left(F(x), \overline{F(x)}\right) - \frac{1}{F(x)} \right| \rightarrow 0.$$

Since  $F \in \text{AP}$  implies  $\overline{F} \in \text{AP}$ , we deduce that  $p_n(F, \overline{F})$  belongs to the algebra  $\text{AP}$ , too. Because  $\text{AP}$  is uniformly closed we obtain  $1/F \in \text{AP}$ .

ii) Let  $\mathbf{F} := (F_1, \dots, F_n) \in U_n(\text{AP})$ . Then  $\mathbf{F}$  is an invertible  $n$ -tuple in  $C_b(\mathbb{R}, \mathbb{C})$ . Hence, there is  $\delta > 0$  such that  $\sum_{j=1}^n |F_j| \geq \delta > 0$ . Conversely, if this latter

condition is satisfied, then the function

$$Q_j := \frac{\overline{F}_j}{\sum_{k=1}^n |F_k|^2}$$

is in  $C_b(\mathbb{R}, \mathbb{C})$ . Since  $F \in \text{AP}$  implies  $\overline{F} \in \text{AP}$ , we see that  $|F|^2 \in \text{AP}$ . Hence  $\sum_{k=1}^n |F_k|^2 \in \text{AP}$ . By i) its inverse belongs to  $\text{AP}$ , as well. Thus  $Q_j \in \text{AP}$ . Since  $\sum_{j=1}^n Q_j F_j = 1$ , we have shown that  $\mathbf{F} \in U_n(\text{AP})$ .  $\square$

**Lemma 2.3.** *Let  $\Lambda$  be a sub-semigroup of  $(\mathbb{R}, +)$ ; that is  $\Lambda$  has the following properties:*

- (i)  $0 \in \Lambda$ ;
- (ii)  $\lambda, \mu \in \Lambda$  implies  $\lambda + \mu \in \Lambda$ ;

Furthermore, let

$$\text{AP}_\Lambda := \{f \in \text{AP} : \sigma(f) \subseteq \Lambda\}.$$

Then

- (1)  $\text{AP}_\Lambda$  is a uniformly closed subalgebra of  $\text{AP}$ .
- (2) If  $\emptyset \neq \Lambda_0 = \{\lambda_1, \dots, \lambda_N\} \subseteq \Lambda \cap \mathbb{R}^+$ , then the evaluation map

$$\Phi_{\Lambda_0} : \begin{cases} A(\mathbb{D}^N) & \rightarrow \text{AP}_\Lambda \\ f & \mapsto \Phi_{\Lambda_0}(f), \end{cases}$$

where  $\Phi_{\Lambda_0}(f)(t) := f(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t})$  is an algebra homomorphism and  $\|\Phi_{\Lambda_0}(f)\|_\infty \leq \|f\|_\infty$ .

- (3) If the positive numbers  $\lambda_1, \dots, \lambda_N$  are linearly independent over  $\mathbb{Q}$ , then  $\Phi_{\Lambda_0}$  is injective and  $\|\Phi_{\Lambda_0}(f)\|_\infty = \|f\|_\infty$ .

*Proof.* (1) We first show that  $\text{AP}_\Lambda$  is uniformly closed. In fact, if  $(f_n)$  is a sequence in  $\text{AP}_\Lambda$  covering uniformly to some  $f \in C_b(\mathbb{R}, \mathbb{C})$ , then  $f \in \text{AP}$  (because  $\text{AP}_\Lambda \subseteq \text{AP}$  and  $\text{AP}$  is closed). Hence, by a standard reasoning,  $\widehat{f}_n(\lambda) \rightarrow \widehat{f}(\lambda)$  for every  $\lambda \in \mathbb{R}$ . Consequently,  $\widehat{f}(\lambda) = 0$  for every  $\lambda \in \mathbb{R} \setminus \Lambda$ ; that is  $f \in \text{AP}_\Lambda$ .

Now we show that  $\text{AP}_\Lambda$  is an algebra. For  $j = 1, 2$ , let  $f_j \in \text{AP}_\Lambda$ . By Theorem 1.5(2), there is a sequence of trigonometric polynomials  $p_n^{(j)}$  with  $\sigma(p_n^{(j)}) \subseteq \sigma(f_j) \subseteq \Lambda$  converging uniformly to  $f_j$ . Now

$$\sigma(p_n^{(1)} + p_n^{(2)}) \subseteq \sigma(p_n^{(1)}) \cup \sigma(p_n^{(2)}) \subseteq \Lambda.$$

Since  $(p_n^{(1)} + p_n^{(2)})$  converges uniformly to  $f_1 + f_2$ , we obtain that  $\sigma(f_1 + f_2) \subseteq \Lambda$ . Hence  $f_1 + f_2 \in \text{AP}_\Lambda$ .

Moreover,

$$\sigma(p_n^{(1)} \cdot p_n^{(2)}) \subseteq [\sigma(p_n^{(1)}) \cup \sigma(p_n^{(2)})] \subseteq \Lambda,$$

where  $[[X]]$  denotes the set  $\{a + b : a, b \in X\}$ . Hence, by a similar reasoning as above,  $f_1 \cdot f_2 \in \text{AP}_\Lambda$ .

Since  $\alpha f \in \text{AP}_\Lambda$  whenever  $f \in \text{AP}_\Lambda$  and  $\alpha \in \mathbb{C}$ , we conclude that  $\text{AP}_\Lambda$  is an algebra over  $\mathbb{C}$ .

(2) This is a consequence of the fact that every  $f \in A(\mathbb{D}^N)$  is the uniform limit of a sequence of polynomials in  $\mathbb{C}[z_1, \dots, z_N]$ .

(3) Using the hypothesis that  $\{\lambda_1, \dots, \lambda_N\}$  is linearly independent over  $\mathbb{Q}$ , we obtain from Kronecker's approximation Theorem 1.1 that

$$E := \{(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t}) : t \in \mathbb{R}\}$$

is dense in  $\mathbb{T}^N$ . Now  $\sup_S |f| = \sup_{\mathbb{T}^N} |f|$  for every dense set  $S$  in  $\mathbb{T}^N$ . Thus, for  $f \in A(\mathbb{D}^N)$ , and  $\mathbf{w} = (w_1, \dots, w_N)$ ,

$$\sup_{t \in \mathbb{R}} |\Phi_\Lambda(f)(t)| = \sup_{t \in \mathbb{R}} |f(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t})| = \sup_{\mathbf{w} \in E} |f(w_1, \dots, w_N)| = \max_{\mathbb{T}^N} |f|.$$

By the distinguished maximum principle, we have for every  $f \in A(\mathbb{D}^N)$  that

$$\max_{\mathbb{T}^N} |f| = \|f\|_\infty.$$

Hence  $\|\Phi_\Lambda(f)\|_\infty = \|f\|_\infty$ . The injectivity of the linear map  $\Phi_\Lambda$  follows.  $\square$

Here is the analogue of Theorem 2.2 for  $AP_\Lambda$  whenever  $\Lambda$  is a subgroup of  $(\mathbb{R}, +)$ .  
<sup>3</sup>

**Proposition 2.4.** *Let  $\Lambda \subseteq \mathbb{R}$  be an additive group. Then  $AP_\Lambda$  is a  $C^*$ -subalgebra of  $C_b(\mathbb{R}, \mathbb{C})$  and the following assertions hold:*

- i) *An element  $F \in AP_\Lambda$  is invertible if and only if  $\delta := \inf_{\mathbb{R}} |F| > 0$ .*
- ii)  *$U_n(AP_\Lambda) = \{(F_1, \dots, F_n) \in (AP_\Lambda)^n : \inf_{\mathbb{R}} \sum_{j=1}^n |F_j| > 0\}$ .*

*Proof.* Since  $\lambda \in \sigma(\bar{f}) \iff -\lambda \in \sigma(f)$ , the assumption “ $\Lambda$  a group” implies that  $\bar{f} \in AP_\Lambda$  whenever  $f \in AP_\Lambda$ . Hence  $AP_\Lambda$  is a  $C^*$ -subalgebra of  $C_b(\mathbb{R}, \mathbb{C})$ . The remaining assertions now follow verbatim as in Theorem 2.2.  $\square$

If  $\Lambda$  merely is a sub-semigroup, the situation is much more difficult. For corona theorems in this setting, see [4] and [5].

For  $\Lambda \subseteq \mathbb{R}$ , let  $[\Lambda]$  denote the  $\mathbb{Q}$ -vector space generated by  $\Lambda$ .

**Lemma 2.5.** *Let  $Q(t) = \sum_{j=1}^N a_j e^{i\lambda_j t}$  be a generalized trigonometric polynomial. Then for every  $\mathbb{Q}$ -linearly independent subset  $\Omega := \{\omega_1, \dots, \omega_M\}$  of  $\Lambda = \{\lambda_1, \dots, \lambda_N\}$  with the property that  $[\Omega] = [\Lambda]$ , there is  $s \in \mathbb{N}^*$ , independent of the coefficients  $a_j$ , and  $q \in C(\mathbb{T}^M, \mathbb{C})$  such that*

$$\Phi_{\frac{\Omega}{s}}(q) = Q,$$

that is

$$q(e^{i(\omega_1/s)t}, \dots, e^{i(\omega_M/s)t}) = Q(t).$$

*Proof.* If  $\Lambda$  itself is  $\mathbb{Q}$ -linearly independent, then the uniquely determined function

$$q(w_1, \dots, w_N) = \sum_{j=1}^N a_j w_j$$

satisfies  $\Phi_\Lambda(q) = Q$  (see Lemma 2.3). Modulo a re-enumeration, let  $\Omega = \{\lambda_1, \dots, \lambda_M\}$  and let  $M < j \leq N$ . Since  $\lambda_j \in [\Omega]$ , there are  $s_j \in \mathbb{N}^*$  and  $s_{n,j} \in \mathbb{Z}$  such that

$$s_j \lambda_j = \sum_{n=1}^M s_{n,j} \lambda_n.$$

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<sup>3</sup> The result itself is not new and we present it only since we could not pin down our proof in the literature.

Let  $s = \prod_{j=M+1}^N s_j$  and  $r_{n,j} := s_{n,j} \prod_{\substack{k=M+1 \\ k \neq j}}^N s_k$ . Then

$$s\lambda_j = \sum_{n=1}^M r_{n,j} \lambda_n, \quad (j = M+1, \dots, N).$$

Hence, whenever  $z_j = e^{i(\lambda_j/s)t}$ , ( $j = 1, \dots, n$ ),

$$\begin{aligned} Q(t) &= \sum_{j=1}^M a_j e^{i\lambda_j t} + \sum_{j=M+1}^N a_j e^{i \sum_{n=1}^M (r_{n,j}/s) \lambda_n t} \\ &= \sum_{j=1}^M a_j e^{i\lambda_j t} + \sum_{j=M+1}^N a_j \prod_{n=1}^M e^{i(\lambda_n/s) r_{n,j} t} \\ &= \sum_{j=1}^M a_j z_j^s + \sum_{j=M+1}^N a_j \prod_{n=1}^M z_{n,j}^{r_{n,j}} \\ &=: q(z_1, \dots, z_M). \end{aligned}$$

We deduce that  $\Phi_{\Omega'}(q) = Q$ , whith  $\Omega' = \{\omega_j/s : j = 1, \dots, M\}$ .  $\square$

### 3. THE STABLE RANKS OF $\text{AP}_\Lambda$

**Definition 3.1.** Let  $A$  be a commutative unital algebra (real or complex) (or just a commutative unital ring) with identity element denoted by 1.

- i) An  $(n+1)$ -tuple  $(f_1, \dots, f_n, g) \in U_{n+1}(A)$  is called *reducible* if there exists  $(a_1, \dots, a_n) \in A^n$  such that  $(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(A)$ .
- (2) The *Bass stable rank* of  $A$ , denoted by  $\text{bsr } A$ , is the smallest integer  $n$  such that every element in  $U_{n+1}(A)$  is reducible. If no such  $n$  exists, then  $\text{bsr } A = \infty$ .

**Definition 3.2.** Let  $A$  be a commutative unital Banach algebra. The *topological stable rank*,  $\text{tsr } A$ , of  $A$  is the least integer  $n$  for which  $U_n(A)$  is dense in  $A^n$ , or infinite if no such  $n$  exists.

It is well known that for Banach algebras,  $\text{bsr } A \leq \text{tsr } A$  and that strict inequality is possible (mostly for Banach algebras of holomorphic functions such as the disk algebra or  $H^\infty(\mathbb{D})$ .) In order to determine the stable ranks of  $\text{AP}_\Lambda$  we move to the polydisk algebra and apply Lemma 2.3.

**Lemma 3.3.**

- (i) Let  $s \in \mathbb{N}^*$  and let  $z^{1/s} = \exp(\frac{1}{s} \log z)$  be the canonical branch of the  $s$ -th root of  $z$  on  $\mathbb{C} \setminus ]-\infty, 0]$ . Let  $r = |z|$  and  $\theta = \arg z$  where  $-\pi < \theta < \pi$ . Then

$$g_s : \begin{cases} 2\overline{\mathbb{D}} \setminus [-2, 0] & \rightarrow \mathbb{T}^2 \\ z & \mapsto (e^{i \frac{\arccos(r/2)}{s}} e^{i\theta/s}, e^{-i \frac{\arccos(r/2)}{s}} e^{i\theta/s}) \end{cases}$$

is a continuous map, where  $\arccos : [-1, 1] \rightarrow [0, \pi]$  is the standard inverse of the cosine function.

(ii) *The map*

$$f_s : \begin{cases} \mathbb{T}^2 & \rightarrow 2\overline{\mathbb{D}} \\ (z_1, z_2) & \mapsto z_1^s + z_2^s \end{cases}$$

*is a continuous surjection such that  $f_s \circ g_s = \text{id}$  on  $2\overline{\mathbb{D}} \setminus [-2, 0]$ .*

The straightforward proof is left to the reader.

Our key to the calculation of the Bass stable rank of  $\text{AP}_\Lambda$  will be the following class of examples of non-reducible tuples in the polydisk algebra. Let us emphasize that non-reducibility in  $A(\mathbb{D}^N)$  will not be sufficient; we need invertible tuples in  $A(\mathbb{D}^N)$  that are non-reducible in  $C(\mathbb{T}^N, \mathbb{C})$ , which is a stronger property. In fact  $(z_1, 1 - z_1 z_2)$  is an invertible tuple that is non-reducible in  $A(\mathbb{D}^2)$  (since  $F(z_1, \bar{z}_1) := z_1 + h(z_1, \bar{z}_1)(1 - |z_1|^2)$  is an extension of the identity map on  $\mathbb{T}$ , and so has a zero in  $\mathbb{D}$ ), but of course,  $F(z_1, z_2) = z_1 + h^*(z_1, z_2)(1 - z_1 z_2) \neq 0$  on  $\mathbb{T} \times \mathbb{T}$  whenever  $h^* \equiv 0$ .

**Lemma 3.4.** *For  $j = 1, \dots, 2N$  and  $s \in \mathbb{N}^*$ , let*

$$f_j(z_1, \dots, z_{4N}) = z_{2j-1}^s + z_{2j}^s - 1,$$

*and*

$$g = \frac{1}{4} - \sum_{j=1}^N f_j f_{N+j}.$$

*Then  $\mathbf{F} := (f_1, \dots, f_N, g)$  is an invertible  $(N+1)$ -tuple in  $A(\mathbb{D}^{4N})$  that is neither reducible in  $A(\mathbb{D}^{4N})$  nor in  $C(\mathbb{T}^{4N}, \mathbb{C})$ .*

*Proof.* It is clear that  $\mathbf{F} \in U_{N+1}(A(\mathbb{D}^{4N}))$ . Let  $\mathbf{h} = (h_1, \dots, h_N) \in C(\mathbb{T}^{4N}, \mathbb{C})^N$  and consider in  $C(\mathbb{T}^{4N}, \mathbb{C})$  the  $N$ -tuple

$$\mathbf{H} := (f_1 + h_1 g, \dots, f_N + h_N g).$$

We claim that  $\mathbf{H}(z_1^{(0)}, \dots, z_{4N}^{(0)}) = \mathbf{0}_N$  for some  $(z_1^{(0)}, \dots, z_{4N}^{(0)}) \in \mathbb{T}^{4N}$ . To this end, we make the following transformations.

Let  $z_{j+2N} = \bar{z}_j$  for  $j = 1, \dots, 2N$ . Then, with  $\boldsymbol{\xi} := (z_1, \dots, z_{2N}, \bar{z}_1, \dots, \bar{z}_{2N})$ ,

$$\mathbf{H}(\boldsymbol{\xi}) = \left( f_1(\boldsymbol{\xi}) + h_1(\boldsymbol{\xi}) \left( \frac{1}{4} - \sum_{j=1}^N |f_j(\boldsymbol{\xi})|^2 \right), \dots, f_N(\boldsymbol{\xi}) + h_N(\boldsymbol{\xi}) \left( \frac{1}{4} - \sum_{j=1}^N |f_j(\boldsymbol{\xi})|^2 \right) \right).$$

Let  $D := \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1/2\}$ . Then  $D \subseteq 2\overline{\mathbb{D}} \setminus [-2, 0]$ . If  $u_{2j-1} \in D$ ,  $j = 1, \dots, N$ , then, using Lemma 3.3 and  $g_s = (G_1, G_2)$ , we put  $z_{2j-1} := G_1(u_{2j-1})$ ,  $z_{2j} := G_2(u_{2j-1})$ , and

$$h_j^*(u_1, u_3, \dots, u_{2N-1}) :=$$

$$h_j \left( G_1(u_1), G_2(u_1), \dots, G_1(u_{2N-1}), G_2(u_{2N-1}), \overline{G_1}(u_1), \overline{G_2}(u_1), \dots, \overline{G_1}(u_{2N-1}), \overline{G_2}(u_{2N-1}) \right).$$

Since  $u_{2j-1} = z_{2j-1}^s + z_{2j}^s$ , it suffices to show that the functions

$$u_{2j-1} - 1 + h_j^*(u_1, u_3, \dots, u_{2N-1}) \left( \frac{1}{4} - \sum_{j=1}^N |u_{2j-1} - 1|^2 \right)$$

have a common zero in  $D^N$ . To do so, let  $w_j := u_{2j-1} - 1$  and put

$$h_j^{**}(w_1, \dots, w_N) := h_j^*(w_1 + 1, \dots, w_N + 1), |w_j| \leq 1/2.$$

Since then unit ball  $\mathbf{B}_N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : \sum_{j=1}^N |z_j|^2 \leq 1\}$  has the property that  $\mathbf{B}_N \subseteq \overline{\mathbb{D}}^N$ , it follows that the functions  $h_j^{**}$  are continuous and bounded on  $(1/2)\mathbf{B}_N$ .

By Brouwer's fixed-point theorem (or no-retract theorem), see for example [9, p. 127], the identity map on  $\partial(1/2)\mathbf{B}_N$  has no zero-free extension to  $(1/2)\mathbf{B}_N$ . Therefore, the map

$$\left( w_1 + h_1^{**}(w_1, \dots, w_N) \left( \frac{1}{4} - \sum_{j=1}^N |w_j|^2 \right), \dots, w_N + h_N^{**}(w_1, \dots, w_N) \left( \frac{1}{4} - \sum_{j=1}^N |w_j|^2 \right) \right)$$

admits a zero  $\Xi := (w_1^{(0)}, \dots, w_N^{(0)})$  in the open ball  $(1/2)\mathbf{B}_N$ . Thus we see that with

$$z_{2j-1}^{(0)} := G_1(w_j^{(0)} + 1), \quad z_{2j}^{(0)} := G_2(w_j^{(0)} + 1), \quad (j = 1, \dots, N)$$

and  $z_{j+2N}^{(0)} = \bar{z}_j^{(0)}$  for  $j = 1, \dots, 2N$ ,  $(z_1^{(0)}, \dots, z_{4N}^{(0)}) \in \mathbb{T}^{4N}$  and

$$H(z_1^{(0)}, \dots, z_{4N}^{(0)}) = \mathbf{0}_N.$$

□

**Theorem 3.5.** *The Bass and topological stable ranks of the Banach algebras  $\text{AP}_\Lambda$  are infinite whenever the dimension of the vector space  $[\Lambda]$  generated by  $\Lambda$  over  $\mathbb{Q}$  is infinite.*

*Proof.* Let  $A = \text{AP}_\Lambda$ . We may suppose, without loss of generality, that  $\Lambda$  contains infinitely many  $\mathbb{Q}$ -linearly independent *positive* reals; otherwise the algebra

$$\text{AP}_{-\Lambda} = \{\bar{f} : f \in \text{AP}_\Lambda\},$$

which is isomorphic isometric to  $\text{AP}_\Lambda$ , has to be considered.

Since  $\text{bsr } A \leq \text{tsr } A$  for any Banach algebra  $A$ , it suffices to show that  $\text{bsr } A = \infty$ . Fix  $N \in \mathbb{N}^*$  and consider the algebras  $C(\mathbb{T}^{4N}, \mathbb{C})$  and  $A(\mathbb{D}^{4N})$ . Let  $\mathbf{a} := (\mathbf{f}, g) := (f_1, \dots, f_N, g)$  be an invertible  $(N+1)$ -tuple in  $A(\mathbb{D}^{4N})$  such that

$$(\tilde{\mathbf{f}}(z_1, \dots, z_{4N}), \tilde{g}(z_1, \dots, z_{4N})) := (\mathbf{f}(z_1^\nu, \dots, z_{4N}^\nu), g(z_1^\nu, \dots, z_{4N}^\nu))$$

is not reducible in  $C(\mathbb{T}^{4N}, \mathbb{C})$  for any  $\nu \in \mathbb{N}^*$  (such a tuple exists by Lemma 3.4). Then  $\tilde{\mathbf{a}} := (\tilde{\mathbf{f}}, \tilde{g})$  is not reducible in  $C(\mathbb{T}^m, \mathbb{C})$  for every  $m \geq 4N$ .

Let  $\{\lambda_1, \dots, \lambda_{4N}\}$  be a set of positive reals in  $\Lambda$  that is independent over  $\mathbb{Q}$ . Let

$$\mathbf{F}(t) := \mathbf{f}(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t}) \text{ and } G(t) := g(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t}).$$

We claim that

$$\mathbf{A}(t) := \mathbf{a}(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t})$$

is an invertible  $(N+1)$ -tuple in  $\text{AP}_\Lambda$  that is not reducible in  $\text{AP}$  (and a fortiori not in  $\text{AP}_\Lambda$ ). Assume for the moment that this has been verified. Then we may conclude that for  $A = \text{AP}$  and  $A = \text{AP}_\Lambda$ ,  $\text{bsr } A \geq N+1$ . Since  $N$  was arbitrarily chosen, we deduce that  $\text{bsr } \text{AP} = \text{bsr } \text{AP}_\Lambda = \infty$ .

Let us introduce the following notation: if  $\mathbf{a}, \mathbf{b} \in A^m$ , then  $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^m a_j b_j$ . To verify the claim, we note that  $\mathbf{a} \cdot \mathbf{b} = 1$  for  $\mathbf{a}, \mathbf{b} \in A(\mathbb{D}^{4N})^{N+1}$  obviously implies by Lemma 2.3 that  $\mathbf{A} \cdot \mathbf{B} = 1$  in  $\text{AP}_\Lambda$ , where

$$\mathbf{B}(t) := \mathbf{b}(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t}).$$



In view of achieving a contradiction, suppose that  $\mathbf{A}$  is reducible in AP. Then there exists  $\mathbf{H} = (H_1, \dots, H_N) \in (\text{AP})^N$  such that

$$\mathbf{F} + \mathbf{H}G \in U_N(\text{AP}).$$

Let  $\mathbf{F} = (F_1, \dots, F_N)$ . Hence, by Theorem 2.2,

$$\sum_{j=1}^N |F_j + H_j G| \geq \delta > 0 \text{ on } \mathbb{R}.$$

For  $j = 1, \dots, N$ , let  $H_j^*(t) = \sum_{k=1}^{M_j} \sigma_{k,j} e^{i\lambda_{k,j}t} \in \text{AP}$  be chosen close to  $H_j(t)$  (uniformly in  $t$ ) so that

$$(3.1) \quad \sum_{j=1}^N |F_j + H_j^* G| \geq \delta/2 > 0 \text{ on } \mathbb{R}$$

(note that  $\text{AP} \subseteq C_b(\mathbb{R}, \mathbb{C})$ ).

By a standard result in linear algebra, there is a subset  $S'$  of

$$S = \{\lambda_{k,j} : k = 1, \dots, M_j, j = 1, \dots, N\}$$

such that the elements in  $\Lambda := S' \cup \{\lambda_1, \dots, \lambda_{4N}\}$  are independent over  $\mathbb{Q}$  and such that

$$[\Lambda] = [\{\lambda_1, \dots, \lambda_{4N}\} \cup S].$$

Let  $L$  be the cardinal of  $\Lambda$ ; that is

$$\Lambda = \{\lambda_1, \dots, \lambda_{4N}, \lambda_{4N+1}, \dots, \lambda_L\}.$$

Then, by Lemma 2.5, there exists  $s \in \mathbb{N}^*$  such that for every  $j \in \{1, \dots, N\}$ ,

$$H_j^* = \Phi_{\frac{1}{s}}(h_j^*)$$

for some function

$$h_j^*(z_1, \dots, z_L) \in C(\mathbb{T}^L, \mathbb{C}),$$

where the evaluation functional is given by

$$\Phi_{\frac{1}{s}}(h)(t) = h(e^{i(\lambda_1/s)t}, \dots, e^{i(\lambda_L/s)t}).$$

Note that by Lemma 2.3,  $\Phi_{\frac{1}{s}}$  is injective. Moreover,

$$\Phi_{\frac{1}{s}}^{-1}(F_j)(z_1, \dots, z_{4N}) = f_j(z_1^s, \dots, z_{4N}^s),$$

as well as

$$\Phi_{\frac{1}{s}}^{-1}(G)(z_1, \dots, z_{4N}) = g(z_1^s, \dots, z_{4N}^s).$$

Since by Kronecker's Theorem 1.1

$$\{(e^{i(\lambda_1/s)t}, \dots, e^{i(\lambda_L/s)t}) : t \in \mathbb{R}\}$$

is dense in  $\mathbb{T}^L$ , we obtain from

$$\sum_{j=1}^N \left| f_j(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t}) + h_j^*(e^{i(\lambda_1/s)t}, \dots, e^{i(\lambda_L/s)t}) g(e^{i\lambda_1 t}, \dots, e^{i\lambda_{4N} t}) \right| \underset{(3.1)}{\geq} \delta/2 > 0$$

that

$$\sum_{j=1}^N |f_j(z_1^s, \dots, z_{4N}^s) + h_j^*(z_1, \dots, z_L) g(z_1^s, \dots, z_{4N}^s)| \geq \delta/2 > 0 \text{ on } \mathbb{T}^L.$$

This tells us that  $(\tilde{f}, \tilde{g})$  is reducible in  $C(\mathbb{T}^L, \mathbb{C})$ ; a contradiction.  $\square$

Without the assumption that  $\dim[\Lambda] = \infty$ , the Bass and topological stable ranks of  $\text{AP}_\Lambda$  may be one: just take  $\Lambda = \mathbb{Z}$ . Then  $\text{AP}_\Lambda$  coincides with the algebra of  $2\pi$ -periodic, continuous functions on  $\mathbb{R}$ , which is isomorphic isometric to  $C(\mathbb{T}, \mathbb{C})$ , and  $\text{bsr } C(\mathbb{T}, \mathbb{C}) = \text{tsr } C(\mathbb{T}, \mathbb{C}) = 1$  (see [17] and [13, p. 8]).

More general, we have the following result:

**Theorem 3.6.** *Suppose that  $\Lambda_0 = \{\lambda_1, \dots, \lambda_N\}$  is a set of  $\mathbb{Q}$ -linearly independent, positive reals. Let*

$$\Lambda_1 := \left\{ \sum_{j=1}^N s_j \lambda_j : s_j \in \mathbb{N} \right\}$$

and

$$\Lambda_2 = \left\{ \sum_{j=1}^N s_j \lambda_j : s_j \in \mathbb{Z} \right\}.$$

Then

$$A_1 := \text{AP}_{\Lambda_1} = \{f \in \text{AP} : \sigma(f) \subseteq \Lambda_1\}$$

is a uniformly closed subalgebra of  $\text{AP}^+$  that is isomorphic isometric to  $A(\mathbb{D}^N)$  and

$$A_2 := \text{AP}_{\Lambda_2} = \{f \in \text{AP} : \sigma(f) \subseteq \Lambda_2\}$$

is a uniformly closed subalgebra of  $\text{AP}$  that is isomorphic isometric to  $C(\mathbb{T}^N, \mathbb{C})$ . In particular,

$$\text{bsr } A_1 = \text{bsr } A(\mathbb{D}^N) = \left\lfloor \frac{N}{2} \right\rfloor + 1, \quad \text{tsr } A_1 = \text{tsr } A(\mathbb{D}^N) = N + 1,$$

$$\text{bsr } A_2 = \text{bsr } C(\mathbb{T}^N, \mathbb{C}) = \left\lfloor \frac{N}{2} \right\rfloor + 1 \text{ and } \text{tsr } A_2 = \text{tsr } C(\mathbb{T}^N, \mathbb{C}) = \left\lfloor \frac{N}{2} \right\rfloor + 1.$$

*Proof.* Let  $\tilde{A}_1 = A(\mathbb{D}^N)$  and  $\tilde{A}_2 = C(\mathbb{T}^N, \mathbb{C})$ . In view of Lemma 2.3, it suffices to show that the evaluation map

$$\Phi_{\Lambda_0} : \begin{cases} \tilde{A}_j & \rightarrow A_j \\ f & \mapsto \Phi_{\Lambda_0}(f), \end{cases}$$

where  $\Phi_{\Lambda_0}(f)(t) := f(e^{i\lambda_1 t}, \dots, e^{i\lambda_N t})$  is a surjection.

Let  $F \in A_1$ . By Theorem 1.5, for  $n \in \mathbb{N}^*$ , there is a trigonometric polynomial  $Q_n$  whose Bohr spectrum  $\sigma(Q_n)$  is contained in  $\Lambda_1$  such that  $\|Q_n - F\|_\infty < 1/n$ . Now

$$Q_n(t) = \sum_{j=1}^M a_j e^{i \sum_{k=1}^N n_{k,j} \lambda_k t} = \sum_{j=1}^M a_j \prod_{k=1}^N e^{i n_{k,j} \lambda_k t},$$

where  $n_{k,j} \in \mathbb{N}$  and  $a_j \in \mathbb{C}$ . The polynomial

$$q_n(z_1, \dots, z_N) := \sum_{j=1}^M a_j \prod_{k=1}^N z_k^{n_{k,j}}$$

now has the property that

$$\Phi_{\Lambda_0}(q_n)(t) = Q_n(t).$$

Since  $\Phi_{\Lambda_0}$  is an isometry (Lemma 2.3), we finally obtain that  $\Phi_{\Lambda_0}(f) = F$ , where  $f$  is the limit point of the Cauchy sequence  $(q_n)$  in  $A(\mathbb{D}^N)$ .

If  $F \in A_2$ , then we use that  $e^{im\lambda_k t} = z_k^m$  whenever  $m \geq 0$  and  $e^{im\lambda_k t} = \overline{z_k}^{|m|}$  whenever  $m < 0$  and proceed in a similar way as above. Recall that  $C(\mathbb{T}^N, \mathbb{C})$  is, by Weierstrass' theorem, the uniform closure of the polynomials in  $z_j$  and  $\overline{z_j}$ .

The remaining assertions follow from the corresponding results in  $A(\mathbb{D}^N)$  and  $C(\mathbb{T}^N, \mathbb{C})$ , since the Bass and topological stable ranks are invariant under isomorphic isometries. Recall that  $\text{bsr } A(\mathbb{D}^N) = \lfloor \frac{N}{2} \rfloor + 1$  by [7, Corollary 3.13],  $\text{tsr } A(\mathbb{D}^N) = N+1$  by [6, Theorem 3.1] and  $\text{bsr } C(\mathbb{T}^N, \mathbb{C}) = \text{tsr } C(\mathbb{T}^N, \mathbb{C}) = \lfloor \frac{N}{2} \rfloor + 1$  by [17] (see also [14, p. 156-157]). Note that the covering dimension of  $\mathbb{T}^N$  is  $N$ .  $\square$

We would like to present the following problem:

**Question 3.7.** Give a characterization of those sub-semigroups  $\Lambda$  of  $(\mathbb{R}, +)$  for which the Bass stable rank of  $\text{AP}_\Lambda$  is finite whenever  $\dim[\Lambda] < \infty$ . What about the case where  $\Lambda = \mathbb{Q}$ ?

#### 4. THE ANALYTIC TRACE $\text{AP}^+$ OF THE ALGEBRA $\text{AP}$

Let  $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$  and  $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im} z > 0\}$  the upper half-plane.

**Definition 4.1.** The analytic trace  $\text{AP}^+$  of  $\text{AP}$  is defined as the uniform closure in  $C_b(\mathbb{R}, \mathbb{C})$  of the set of all functions of the form

$$Q(t) = \sum_{j=1}^n a_j e^{i\lambda_j t},$$

where  $a_j \in \mathbb{C}$ ,  $\lambda_j \in \mathbb{R}^+$ , and  $n \in \mathbb{N}^*$ .<sup>4</sup>

Moreover, let  $\text{AP}_{\text{hol}}^+$  denote the uniform closure in  $C_b(\mathbb{C}^+, \mathbb{C})$  of the set of all functions of the form

$$q(z) = \sum_{j=1}^n a_j e^{i\lambda_j z},$$

where  $a_j \in \mathbb{C}$ ,  $\lambda_j \in \mathbb{R}^+$ , and  $n \in \mathbb{N}^*$ .

Note that the only difference in the definitions of the classes  $\text{AP}$  and  $\text{AP}^+$  is that we allow here only non-negative exponents  $\lambda_j$ . The following relations between  $\text{AP}^+$  and  $\text{AP}_{\text{hol}}^+$  are easy to check.

**Theorem 4.2.**

- (1)  $\text{AP}_{\text{hol}}^+$  is a closed subalgebra of  $H^\infty(\mathbb{C}^+)$ .
- (2) Every function  $f \in \text{AP}_{\text{hol}}^+$  has a continuous extension,  $f^*$ , to the boundary  $\mathbb{R}$  of  $\mathbb{C}^+$ .
- (3)  $f^* \in \text{AP}^+$  and  $\|f\|_{\mathbb{C}^+} := \sup_{z \in \mathbb{C}^+} |f(z)| = \|f^*\|_\infty$ .
- (4)  $\text{AP}^+$  is isomorphic isometric to  $\text{AP}_{\text{hol}}^+$ .
- (5) If  $g \in \text{AP}^+$ , then its Poisson-integral

$$[g](z) := \int_{\mathbb{R}} P_y(x-t)g(t) dt, \quad z = x + iy \in \mathbb{C}^+$$

belongs to  $\text{AP}_{\text{hol}}^+$  and  $[g]^* = g$ .

- (6) The Poisson operator  $\text{AP} \rightarrow C(\mathbb{C}^+, \mathbb{C}), f \mapsto [f]$  is multiplicative on  $\text{AP}^+$ .

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<sup>4</sup> Note that this definition coincides with that given in the introduction in view of Theorem 1.5(2).

The following result, that appears with an entirely different proof in [5] does not seem to be widely known.

**Theorem 4.3.** *If  $f \in H^\infty(\mathbb{C}^+)$  has a continuous extension  $f^*$  to  $\mathbb{R}$  such that  $f^* \in \text{AP}$ , then  $f^* \in \text{AP}^+$  and  $f \in \text{AP}_{\text{hol}}^+$ .*

*Proof. Step 1* We first show that for every  $\lambda < 0$  the Fourier-Bohr coefficients

$$\widehat{f^*}(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f^*(t) e^{-i\lambda t} dt$$

of  $f^*$  are zero.

Because  $\lambda < 0$ , the function  $F(z) := f(z)e^{-i\lambda z}$  is bounded on  $\mathbb{C}^+$ . Hence  $F$  belongs to  $H^\infty(\mathbb{C}^+)$  and has a continuous extension to  $\mathbb{R}$ . If  $M$  is an upper bound for  $|f|$ , then

$$|F| \leq M e^{\lambda \text{Im } z} \leq M.$$

For  $T > 0$ , let  $\Gamma_T$  be the boundary of the half disk

$$\{w \in \mathbb{C} : |w| \leq T, \text{Im } w \geq 0\}.$$

By Cauchy's integral theorem,

$$\int_{\Gamma_T} F(\xi) d\xi = 0.$$

A splitting of the curve  $\Gamma_T$  into the upper half circle  $C_T$  and the segment  $[-T, T]$  yields

$$(4.1) \quad 0 = \int_{C_T} F(\xi) d\xi + \int_{-T}^T F(t) dt.$$

We claim that

$$I_T := \frac{1}{2T} \int_{C_T} F(\xi) d\xi \rightarrow 0 \text{ as } T \rightarrow \infty.$$

In fact, due to the symmetry of the sine-function, and the facts that for  $0 \leq \theta \leq \pi/2$  and  $\xi = T e^{i\theta}$

$$|F(\xi)| \leq M e^{T\lambda \sin \theta} \leq M e^{-T|\lambda| \frac{2}{\pi} \theta},$$

we obtain

$$\begin{aligned} |I_T| &\leq \frac{2M}{2T} \int_0^{\pi/2} e^{-T|\lambda| \frac{2}{\pi} \theta} T d\theta \\ &= \frac{M}{T} \frac{1 - e^{-T|\lambda|}}{|\lambda| \frac{2}{\pi}} \rightarrow 0 \text{ as } T \rightarrow \infty. \end{aligned}$$

Hence, by (4.1),

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T F(t) dt = 0.$$

**Step 2** By Step 1, the Bohr spectrum

$$\sigma(f^*) = \{\lambda \in \mathbb{R} : \widehat{f^*}(\lambda) \neq 0\}$$

of  $f^* \in \text{AP}$  belongs to  $\mathbb{R}^+$ . Hence, by Theorem 1.5, we obtain a sequence of trigonometric polynomials

$$q_n(t) = \sum_{j=1}^{N(n)} a_{j,n} e^{i\lambda_{j,n}t}$$

with  $\lambda_{j,n} \geq 0$  such that  $\|q_n - f^*\| \rightarrow 0$ . Thus  $f^* \in \text{AP}^+$ .  $\square$

**Corollary 4.4.**  $\text{AP}^+$  is the set of functions in  $\text{AP}$  that admit a bounded holomorphic extension to  $\mathbb{C}^+$ .

*Proof.* By Theorem 4.2, every  $\text{AP}^+$ -function admits a bounded holomorphic extension to  $\mathbb{C}^+$ . If, on the other hand,  $F$  is a bounded holomorphic extension to  $\mathbb{C}^+$  of  $f \in \text{AP}$ , then Theorem 4.3 implies that  $f = F^* \in \text{AP}^+$ .  $\square$

As a corollary to the main theorem 3.5 we have

**Theorem 4.5.**  $\text{bsr AP}^+ = \text{tsr AP}^+ = \text{bsr AP}_{\text{hol}}^+ = \text{tsr AP}_{\text{hol}}^+ = \infty$ .

## 5. GENERAL SUBALGEBRAS OF $\text{AP}$

The most general result we obtain is the following:

**Theorem 5.1.** Let  $\Lambda = \{\lambda_1, \lambda_2, \dots\}$  be a countably infinite subset of reals that is linearly independent over  $\mathbb{Q}$ . Associate with  $\Lambda$  the following functions:

$$F_j(t) := e^{i\lambda_{2j-1}t} + e^{i\lambda_{2j}t} - 1, \quad j = 1, 2, \dots$$

Then  $F_j \in \text{AP}$  and the Bass stable rank of any (complex) subalgebra  $A$  of  $\text{AP}$  containing the functions 1 and  $F_j$  ( $j \in \mathbb{N}^*$ ), is infinite. Moreover, the tuple  $(F_1, \dots, F_N)$  cannot be approximated in the supremum norm by tuples invertible in  $A$ .<sup>5</sup>

*Proof.* (i) By Definition of  $\text{AP}$ , it is obvious that  $F_j \in \text{AP}$ . Note also that 1 is the identity element in  $A$ . Let

$$G = \frac{1}{4} - \sum_{j=1}^N F_j F_{N+j}.$$

Then  $G \in A$ . Since  $\sum_{j=1}^N F_{N+j} F_j + G = 1/4 \in A^{-1}$ , we see that  $(F_1, \dots, F_N, G)$  is an invertible  $(N+1)$ -tuple in  $A$ . By (the proof of) Theorem 3.5,  $(F_1, \dots, F_N, G)$  is not reducible in  $\text{AP}$ . Consequently, since  $A \subseteq \text{AP}$ ,  $(F_1, \dots, F_N, G)$  is not reducible in  $A$  either. Thus  $\text{bsr } A \geq N+1$ . Since  $N$  was arbitrarily chosen,  $\text{bsr } A = \infty$ .

(ii) To prove the second assertion, suppose that  $(F_1, \dots, F_N)$  does admit approximations by invertible  $N$ -tuples in  $A$ ; say  $\|F_j - H_j\|_\infty < 1/(24N)$  for some  $(H_1, \dots, H_N) \in U_N(A)$ . Since  $U_n(A) \subseteq U_n(\text{AP})$  for every  $n$ , we conclude from  $\|F_j\|_\infty \leq 3$  and

$$\sum_{j=1}^N F_j F_{j+N} + G = \frac{1}{4}$$

---

<sup>5</sup>The usual results on the topological stable rank do not apply, since these algebras  $A$  are not necessarily complete or  $Q$ -algebras (i.e.  $A^{-1}$  open).

that

$$F := \sum_{j=1}^N H_j F_{j+N} + G \in U_1(\text{AP}),$$

because

$$|4F - 1| = 4 \left| \sum_{j=1}^N (H_j - F_j) F_{j+N} \right| \leq 4N \frac{1}{24N} 3 = \frac{1}{2}.$$

Let  $G_j \in \text{AP}$  be chosen so that  $1 = \sum_{j=1}^N H_j G_j$ . Then  $F$  may be rewritten as

$$\begin{aligned} F &= \sum_{j=1}^N H_j F_{j+N} + \sum_{j=1}^N H_j G_j G \\ &= \sum_{j=1}^N H_j (F_{j+N} + G_j G). \end{aligned}$$

Thus  $(F_{N+1}, \dots, F_{N+N}, G)$  is reducible in  $\text{AP}$ ; a contradiction to (i) (note that  $(F_1, \dots, F_N)$  plays the same role as  $(F_{N+1}, \dots, F_{N+N})$  due to the similarity in the definitions of these functions).  $\square$

To sum up, under the assumptions of Theorem 5.1, we were able to determine the Bass stable ranks of all standard subalgebras of  $\text{AP}$  and  $\text{AP}^+$  without using corona-type theorems characterizing the invertible tuples in advance.

As a final important example to which our theory applies, we mention the Wiener-type algebra

$$\text{APW}^+ := \left\{ F(t) := \sum_{j=1}^{\infty} a_j e^{i\lambda_j t} : \|F\| := \sum_{j=1}^{\infty} |a_j| < \infty, \lambda_j \geq 0 \right\};$$

see [11] for many other examples.

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